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Discrete Mathematics 286 (2004) 195 – 202
DISCRETE MATHEMATICS
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Total domination subdivision numbers of trees

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Received 1 May 2003; received in revised form 12 December 2003; accepted 25 June 2004
Available online 21 August 2004

Abstract

A set S of vertices in a graph G is a total dominating set of G if every vertex is adjacent to a vertex in S. The total domination number $\gamma_t(G)$ is the minimum cardinality of a total dominating set of G. The total domination subdivision number $\mathrm{sd}_{\gamma_t}(\mathrm{G})$ of a graph G is the minimum number of edges that must be subdivided (where each edge in G can be subdivided at most once) in order to increase the total domination number. Haynes et al. (J. Combin. Math. Combin. Comput. 44 (2003) 115) showed that for any tree T of order at least 3, $1 \leqslant \mathrm{sd}_{\gamma_t}(\mathrm{T}) \leqslant 3$. In this paper, we give a constructive characterization of trees whose total domination subdivision number is 3.

MSC: 05C69

Keywords: Trees; Total domination number; Total domination subdivision number

1. Introduction

In this paper, we continue the study of the total domination subdivision numbers started by Haynes et al. [5].

Total domination in graphs was introduced by Cockayne et al. [1] and is now well studied in graph theory. The literature on domination in graphs has been surveyed and detailed in the two books by Haynes et al. [3,4].

Let G be a graph with no isolated vertex. If $S, T \subseteq V(G)$ and every vertex of T is adjacent to a vertex of S (other than itself), then we say that S *totally dominates* T. In particular, if $T = V(G)$, then we call S a *total dominating set* (TDS) of G. The *total domination number* of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a TDS. A TDS of G of cardinality $\gamma_t(G)$ is called a $\gamma_t(G)$*-set*.

Haynes et al. [5] define the *total domination subdivision number* $\mathrm{sd}_{\gamma_t}(\mathrm{G})$ of a graph G to be the minimum number of edges that must be subdivided (where each edge in G can be subdivided at most once) in order to increase the total domination number. We assume that the graph G is of order at least three since the total domination number of the graph K_2 does not change when its only edge is subdivided. We remark that the domination subdivision number, defined by Arumugan, has been studied in [2] and elsewhere.

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[1] Research supported in part by the University of KwaZulu-Natal and the South African National Research Foundation.

0012-365X/$ - see front matter
doi:10.1016/j.disc.2004.06.004

For notation and graph theory terminology we in general follow [3]. Specifically, for a vertex v in a rooted tree T, we let $C(v)$ and $D(v)$ denote the set of children and descendants, respectively, of v, and we define $D[v] = D(v) \cup \{v\}$. The *maximal subtree* at v is the subtree of T induced by $D[v]$, and is denoted by T_v. A *leaf* of T is a vertex of degree 1, while a *support vertex* of T is a vertex adjacent to a leaf.

A *cycle* on n vertices is denoted by C_n and a *path* on n vertices by P_n. A *caterpillar* is a tree with the property that the removal of its leaves results in a path, called the *spine* of the caterpillar. The *code* C of a caterpillar T with spine $v_1, v_2, \ldots, v_s$ is the sequence of nonnegative integers $(t_1, t_2, \ldots, t_s)$ where t_i is the number of leaves adjacent to v_i in T. The substrings of consecutive zeros in C are called the *zero strings* of C and are labelled from 1 to k. For $i = 1, 2, \ldots, k$, the number of zeros in string i is denoted by z_i. For example, the caterpillar with code $(1, 0, 0, 0, 1, 0, 1, 0, 0, 1)$ has $z_1 = 3$, $z_2 = 1$, and $z_3 = 2$.

2. Known results

The total domination number of a cycle or a path is easy to compute.

Proposition 1. *For* $n \geqslant 3$, $\gamma_t(C_n) = \gamma_t(P_n) = n/2$ *if* $n \equiv 0 \pmod 4$ *and* $\gamma_t(C_n) = \gamma_t(P_n) = \lfloor n/2 \rfloor + 1$ *otherwise*.

An immediate consequence of Proposition 1 now follows.

Proposition 2. *For a cycle* C_n *and a path* P_n *on* $n \geqslant 3$ *vertices*,

$$\mathrm{sd}_{\gamma_t}(\mathrm{C_n}) = \mathrm{sd}_{\gamma_t}(\mathrm{P_n}) = \begin{cases} 1 & \text{if } n \equiv 0, 1 \pmod 4, \\ 2 & \text{if } n \equiv 3 \pmod 4, \\ 3 & \text{if } n \equiv 2 \pmod 4. \end{cases}$$

Haynes et al. [5] showed that the total domination subdivision number of a tree is either 1, 2, or 3, and so trees can be classified as Class 1, Class 2, or Class 3 depending on whether their total domination subdivision number is 1, 2 or 3, respectively.

Theorem 3 (*Haynes et al. [5]*). *For any tree T of order at least* 3, $1 \leqslant \mathrm{sd}_{\gamma_t}(\mathrm{T}) \leqslant 3$.

The following property of trees in Class 3 is established in [5].

Lemma 4 (*Haynes et al. [5]*). *If T is a tree in Class* 3, *then any two leaves of T are at distance at least five apart*.

In [5] caterpillars in Class 3 are characterized.

Theorem 5 (*Haynes et al. [5]*). *A caterpillar with code C is in Class* 3 *if and only if C has no entry greater than* 1, *no consecutive nonzero entries, and* $z_i \equiv 2 \pmod 4$ for $1 \leqslant i \leqslant k$.

3. Main result

Our aim in this paper is to provide a constructive characterization of all trees in Class 3. For this purpose, we describe a procedure to build a family $\mathscr{F}$ of labelled trees that are of Class 3 as follows. The label of a vertex is also called its *status*, denoted $\mathrm{sta}(v)$. Let $\mathscr{F}$ be the family of labelled trees that:

(i) contains P_6 where the two leaves have status C, the two support vertices have status B, and the two central vertices have status A; and
(ii) is closed under the two operations $\mathscr{T}_1$ and $\mathscr{T}_2$, which extend the tree T by attaching a tree to the vertex $y \in V(T)$, called the *attacher*.

- **Operation** $\mathscr{T}_1$. Assume $\mathrm{sta}(y) = A$. Then add a path x, w, v and the edge xy. Let $\mathrm{sta}(x) = A$, $\mathrm{sta}(w) = B$, and $\mathrm{sta}(v) = C$.
- **Operation** $\mathscr{T}_2$. Assume $\mathrm{sta}(y) \in \{B, C\}$. Then add a path x, w, v, u and the edge xy. Let $\mathrm{sta}(x) = \mathrm{sta}(w) = A$, $\mathrm{sta}(v) = B$, and $\mathrm{sta}(u) = C$.

The two operations $\mathscr{T}_1$ and $\mathscr{T}_2$ are illustrated in Fig. 1.

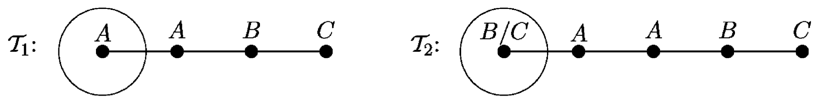

Fig. 1. The two operations.

We now present our main result, a proof of which is presented in Subsections 5.1 and 5.2.

Theorem 6. *A tree T is in Class* 3 *if and only if* $T \in \mathscr{F}$.

The subfamily of caterpillars in the family $\mathscr{F}$ are all constructed from P_6 by repeated applications of operation $\mathscr{T}_2$. These caterpillars are precisely those described in the statement of Theorem 5. Hence, Theorem 5 is an immediate consequence of Theorem 6.

We shall show (see, Subsection 5.1) that the total domination number of each tree in the family $\mathscr{F}$ is even. Hence, as an immediate consequence of Theorem 6 we have the following result:

Corollary 7. *If T is a tree of order at least* 3 *whose total domination number is odd, then T is in Class* 1 *or Class* 2.

4. The family $\mathscr{F}$

If $T \in \mathscr{F}$, we let $A(T)$, $B(T)$, and $C(T)$ be the sets of vertices of status *A*, *B*, and *C*, respectively, in *T*. The following observation follows immediately from the way in which each tree in the family $\mathscr{F}$ is constructed.

Observation 8. *Let* $T \in \mathscr{F}$ *and let* $v \in V(T)$.

(a) *If* $sta(v) = A$, *then v is adjacent to exactly one vertex of* $B(T) \cup C(T)$ *and at least one vertex of* $A(T)$.
(b) *If* $sta(v) = B$ (*respectively*, $sta(v) = C$), *then v is adjacent to exactly one vertex, say x, of status C* (*respectively, B*). *Moreover*, $N(v) - \{x\} \subseteq A(T)$.
(c) *If v is a support vertex, then* $sta(v) = B$.
(d) *If v is a leaf, then* $sta(v) = C$.
(e) *If v is a vertex at distance* 2 *from a vertex in* $B(T)$ (*respectively, in* $C(T)$), *then* $sta(v) = A$.
(f) $|B(T)| = |C(T)|$.

5. Proof of Theorem 6

In this section, we present a proof of our main result, namely Theorem 6.

5.1. Sufficiency of Theorem 6

We proceed with a proof of the sufficiency condition in Theorem 6 by first presenting the following three lemmas.

Lemma 9. *If* $T \in \mathscr{F}$, *then* $\gamma_t(T) = |B(T)| + |C(T)| = 2|B(T)|$ *and* $B(T) \cup C(T)$ *is a* $\gamma_t(T)$*-set. Moreover, if T is obtained from* T' *using operation* $\mathscr{T}_1$ *or* $\mathscr{T}_2$, *then* $\gamma_t(T) = \gamma_t(T') + 2$.

Proof. By Observation 8, $B(T) \cup C(T)$ is a TDS of *T* implying that $\gamma_t(G) \leqslant |B(T)| + |C(T)| = 2|B(T)|$.

To see that $\gamma_t(T) \geqslant 2|B(T)|$, let *S* be a $\gamma_t(T)$-set. If $S = B(T) \cup C(T)$, then we are finished. Hence assume that *X* is the set of vertices of $B(T) \cup C(T)$ that are in $V - S$. We show that for each $x \in X$ we can associate a unique vertex $a \in S \cap A(T)$. If *x* is adjacent to a vertex, say *a*, in $S \cap A(T)$, then Observation 8(a) implies that *a* has no other neighbor in *X*. Moreover, *a* has no neighbor in $S - A(T)$. Thus, for each such *x*, we can associate a unique $a \in S \cap A(T)$. If *x* is not adjacent to a vertex in $S \cap A(T)$, then *x* must be dominated by a vertex, say *w*, in $S - A(T)$. Since $w \in B(T) \cup C(T)$, Observation 8(b) shows that *x* is the only neighbor of *w* in *X* (and *w* is the only neighbor of *x* in *S*) and *w* has no neighbor in $S - A(T)$. Since *S* is a total dominating set, *w* must have a neighbor, say *a*, in $S \cap A(T)$. Again by Observation 8(a), the vertex *a* is not adjacent to any vertex except *w* from $X \cup (S - A(T))$.

Hence, a is not associated with another vertex of X and we may associate it with x. Thus, for each $x \in X$, we have associated a unique vertex in $S \cap A(T)$. It follows that $\gamma_t(T) = |S| = |A(T) \cap S| + |S - A(T)| \geqslant |X| + |S - A(T)| = |B(T) \cup C(T)| = 2|B(T)|$. Hence, $\gamma_t(T) = 2|B(T)|$ and $B(T) \cup C(T)$ is a $\gamma_t(T)$-set.

If $T \in \mathscr{F}$ is obtained from T' using operation $\mathscr{T}_1$ or $\mathscr{T}_2$, then T has exactly two more vertices with status B or C than T' has. Since $\gamma_t(T) = |B(T)| + |C(T)|$ and $\gamma_t(T') = |B(T')| + |C(T')|$, it follows that $\gamma_t(T) = \gamma_t(T') + 2$. □

Lemma 10. *If $T \in \mathscr{F}$, then*

(a) *every vertex of T is in some $\gamma_t(T)$-set, and*
(b) *if $v \in B(T) \cup C(T)$, there exists a set S containing v that totally dominates $V(T) - \{v\}$ such that $|S| = \gamma_t(T) - 1$.*

Proof. The tree T can be obtained from a sequence $T_1, \ldots, T_m$ of trees where T_1 is a path P_6 and $T = T_m$, and, if $m \geqslant 2$, T_{i+1} can be obtained from T_i by operation $\mathscr{T}_1$ or $\mathscr{T}_2$ for $i = 1, \ldots, m-1$. To prove the desired result, we proceed by induction on m.

If $m = 1$, then $T = P_6$, and (a) and (b) are immediate. Thus, $\gamma_t(T) = 4$ and every vertex of T is in some $\gamma_t(T)$-set. Suppose T is the path a, b, c, d, e, f where $B(T) = \{b, e\}$ and $C(T) = \{a, f\}$. For $v \in \{a, b\}$, $S = \{v, d, e\}$ is a set containing v that totally dominates $V(T) - \{v\}$ such that $|S| = \gamma_t(T) - 1$. By symmetry we have such sets for e and f. This establishes the base case. Assume, then, that the result holds for all trees in $\mathscr{F}$ that can be constructed from a sequence of fewer than m trees, where $m \geqslant 2$. Let $T \in \mathscr{F}$ be obtained from a sequence $T_1, T_2, \ldots, T_m$ of m trees. For notational convenience, we denote T_{m-1} simply by T'. By our inductive hypothesis, every vertex of T' is in some $\gamma_t(T')$-set. By Lemma 9, $\gamma_t(T) = \gamma_t(T') + 2$. We now consider two possibilities depending on whether T is obtained from T' by operation $\mathscr{T}_1$ or $\mathscr{T}_2$.

Case 1: T is obtained from T' by operation $\mathscr{T}_2$.

Suppose T is obtained from T' by adding a path y, x, w, v, u of length 4 to the attacher vertex $y \in V(T')$. Then, $\mathrm{sta}(y) \in \{B, C\}$, $\mathrm{sta}(x) = \mathrm{sta}(w) = A$, $\mathrm{sta}(v) = B$, and $\mathrm{sta}(u) = C$. For any $\gamma_t(T')$-set S', $S' \cup \{w, v\}$ is a $\gamma_t(T)$-set. Hence every vertex of $V(T') \cup \{w, v\}$ is in some $\gamma_t(T)$-set. By our inductive hypothesis, there is a set $D' \subset V(T')$ containing y such that $|D'| = \gamma_t(T') - 1$ and D' totally dominates $V(T') - \{y\}$. Then, $D' \cup \{u, v, x\}$ is a $\gamma_t(T)$-set. Hence every vertex of T is in some $\gamma_t(T)$-set. Furthermore, $D' \cup \{v, x\}$ (respectively, $D' \cup \{u, x\}$) is a set of cardinality $\gamma_t(T) - 1$ containing v (respectively, u) that totally dominates $V(T) - \{v\}$ (respectively, $V(T) - \{u\}$). Moreover, for any $v' \in B(T') \cup C(T')$, let $D \subset V(T')$ be a set containing v' such that $|D| = \gamma_t(T') - 1$ and D totally dominates $V(T') - \{v'\}$. It follows that $D \cup \{v, w\}$ is a set with cardinality $\gamma_t(T) - 1$ that totally dominates $V(T) - \{v'\}$. Hence, the lemma holds.

Case 2: T is obtained from T' by operation $\mathscr{T}_1$.

The proof of Case 2 is very similar to that of the previous case and is therefore omitted. □

Lemma 11. *If $T \in \mathscr{F}$ and T^* is obtained from T by subdividing one edge, then $\gamma_t(T^*) = \gamma_t(T)$. Furthermore, there exists a $\gamma_t(T^*)$-set containing any specified vertex in $B(T) \cup C(T)$.*

Proof. Let e be the edge of T that is subdivided to form T^*. Clearly, $\gamma_t(T) \leqslant \gamma_t(T^*)$. As in the first paragraph of Lemma 10, we proceed by induction on the length m of the sequence of trees needed to construct the tree T.

The base case when $m = 1$ is immediate. Assume, then, that the result holds for all trees in $\mathscr{F}$ that can be constructed from a sequence of fewer than m trees, where $m \geqslant 2$. Let $T \in \mathscr{F}$ be obtained from a sequence $T_1, T_2, \ldots, T_m$ of m trees. For notational convenience, we denote T_{m-1} simply by T'. By our inductive hypothesis, if T'' is obtained from T' by subdividing one edge, then $\gamma_t(T'') = \gamma_t(T')$ and there exists a $\gamma_t(T'')$-set containing any specified vertex in $B(T') \cup C(T')$. We now consider two possibilities depending on whether T is obtained from T' by operation $\mathscr{T}_1$ or $\mathscr{T}_2$.

Case 1: T is obtained from T' by operation $\mathscr{T}_1$.

Suppose T is obtained from T' by adding a path y, x, w, v of length 3 to the attacher vertex $y \in V(T')$. Then, $\mathrm{sta}(y) = \mathrm{sta}(x) = A$, $\mathrm{sta}(w) = B$, and $\mathrm{sta}(v) = C$.

Case 1.1: $e \in E(T')$.

Let T'' be obtained from T' by subdividing the edge e. Thus, T^* is obtained from T'' by adding the path y, x, w, v to the vertex $y \in V(T'')$. Any $\gamma_t(T'')$-set can be extended to a TDS of T^* by adding to it the vertices v and w, and so $\gamma_t(T^*) \leqslant \gamma_t(T'') + 2$. By the inductive hypothesis, $\gamma_t(T'') = \gamma_t(T')$ and by Lemma 9, $\gamma_t(T') = \gamma_t(T) - 2$. Hence, $\gamma_t(T^*) - 2 \leqslant \gamma_t(T'') = \gamma_t(T') = \gamma_t(T) - 2 \leqslant \gamma_t(T^*) - 2$. Consequently, we must have equality throughout this inequality chain. In particular, $\gamma_t(T^*) = \gamma_t(T)$ and there is a $\gamma_t(T^*)$-set containing w and v. Further if $z \in B(T') \cup C(T')$, then, by the inductive hypothesis, there exists a $\gamma_t(T'')$-set S' containing z, and so $S' \cup \{v, w\}$is a $\gamma_t(T^*)$-set containing z.

Case 1.2: $e \in E(T) - E(T')$.

By Lemma 10, there is a $\gamma_t(T')$-set S' containing y.

Suppose $e \neq vw$. Then, $S' \cup \{v, w\}$ is a TDS of T^*, and so $\gamma_t(T) \leqslant \gamma_t(T^*) \leqslant \gamma_t(T') + 2 = \gamma_t(T)$. Consequently, $\gamma_t(T^*) = \gamma_t(T)$ and there is a $\gamma_t(T^*)$-set containing v and w. Let $z \in B(T') \cup C(T')$. Then, by Lemma 10, there exists a $\gamma_t(T')$-set

D' containing z. Adding to D' the vertex w and its neighbor of degree two in T^* produces a $\gamma_t(T^*)$-set containing z.

Suppose $e = vw$. Let c denote the resulting vertex of degree 2 in T^* when the edge e is subdivided. Then, $S' \cup \{c, w\}$ and $S' \cup \{c, v\}$ are both TDS of T^*, and so as before $\gamma_t(T^*) = \gamma_t(T)$ and each of v and w is contained in a $\gamma_t(T^*)$-set. Let $z \in B(T') \cup C(T')$. Then, with D' as defined in the previous paragraph, $D' \cup \{c, w\}$ is a $\gamma_t(T^*)$-set containing z.

Case 2: T is obtained from T' by operation $\mathscr{T}_2$.

Suppose T is obtained from T' by adding a path y, x, w, v, u of length 4 to the attacher vertex $y \in V(T')$. Then, $\mathrm{sta}(y) \in \{B, C\}$, $\mathrm{sta}(x) = \mathrm{sta}(w) = A$, $\mathrm{sta}(v) = B$, and $\mathrm{sta}(u) = C$.

Case 2.1: $e \in E(T')$.

Let T'' be obtained from T' by subdividing the edge e. Thus, T^* is obtained from T'' by adding the path y, x, w, v, u to the vertex $y \in V(T'')$. Any $\gamma_t(T'')$-set can be extended to a TDS of T^* by adding to it the vertices v and w, and so $\gamma_t(T^*) \leqslant \gamma_t(T'') + 2$. Hence, as in Case 1.1, $\gamma_t(T^*) = \gamma_t(T)$. If $z \in B(T') \cup C(T')$, then, by the inductive hypothesis, there exists a $\gamma_t(T'')$-set S' containing z, and so $S' \cup \{v, w\}$ is a $\gamma_t(T^*)$-set containing z. Further, if $y = z$, then $S' \cup \{u, v\}$ is a $\gamma_t(T^*)$-set. Hence there exists a $\gamma_t(T^*)$-set containing any specified vertex in $B(T) \cup C(T)$.

Case 2.2: $e \in E(T) - E(T')$.

Since $y \in B(T') \cup C(T')$, by Lemma 10, there exists a set S' containing y that totally dominates $V(T') - \{y\}$ such that $|S'| = \gamma_t(T') - 1$. Let c denote the resulting vertex of degree 2 in T^* when the edge e is subdivided. Suppose $e = xy$ (and so, x and y are the two neighbors of c in T^*). Then, $S' \cup \{c, u, v\}$ is a TDS of T^*, and so $\gamma_t(T) \leqslant \gamma_t(T^*) \leqslant \gamma_t(T') + 2 = \gamma_t(T)$. Consequently, $\gamma_t(T^*) = \gamma_t(T)$ and there is a $\gamma_t(T^*)$-set containing u and v. Let $z \in B(T') \cup C(T')$. By Lemma 9, $B(T') \cup C(T')$ is a $\gamma_t(T')$-set. Hence, $B(T') \cup C(T') \cup \{v, w\}$ is a $\gamma_t(T^*)$-set containing z. Similarly, the desired result follows for the other three possible choices for the edge e. □

We are now in a position to prove the sufficiency condition in Theorem 6.

Lemma 12. *If $T \in \mathscr{F}$, then T is in Class* 3.

Proof. As in the first paragraph of Lemma 10, we proceed by induction on the length m of the sequence of trees needed to construct the tree T. Suppose $m = 1$. Then, $T = P_6$ and by Proposition 2, T is in Class 3. This establishes the base case. Assume then that the result holds for all trees in $\mathscr{F}$ that can be constructed from a sequence of fewer than m trees, where $m \geqslant 2$. Let $T \in \mathscr{F}$ be obtained from a sequence $T_1, T_2, \ldots, T_m$ of m trees. For notational convenience, we denote T_{m-1} simply by T'. By our inductive hypothesis, T' is in Class 3.

Let T^* be obtained from T by subdividing any two edges, say e and f, of T. Clearly, $\gamma_t(T) \leqslant \gamma_t(T^*)$. To show that T is in Class 3 it suffices to show that $\gamma_t(T) \geqslant \gamma_t(T^*)$. We now consider two possibilities depending on whether T is obtained from T' by operation $\mathscr{T}_1$ or $\mathscr{T}_2$.

Case 1: T is obtained from T' by operation $\mathscr{T}_1$.

Suppose T is obtained from T' by adding a path y, x, w, v of length 3 to the attacher vertex $y \in V(T')$.

Case 1.1: $e, f \in E(T')$.

Let T'' be obtained from T' by subdividing the edges e and f. Thus, T^* is obtained from T'' by adding the path y, x, w, v to the vertex $y \in V(T'')$. Any $\gamma_t(T'')$-set can be extended to a TDS of T^* by adding to it the vertices v and w, and so $\gamma_t(T^*) \leqslant \gamma_t(T'') + 2$. By the inductive hypothesis, T' is in Class 3, and so $\gamma_t(T'') = \gamma_t(T')$. Hence, by Lemma 9, $\gamma_t(T) = \gamma_t(T') + 2 = \gamma_t(T'') + 2 \geqslant \gamma_t(T^*)$, and the desired result follows.

Case 1.2: $|\{e, f\} \cap E(T')| = 1$.

We may assume $e \in E(T')$ and $f \in E(T) - E(T')$. Let T'' be obtained from T' by subdividing the edge e. Thus, T^* is obtained from T'' by adding the path y, x, w, v to the vertex $y \in V(T'')$ and then subdividing the edge f. Any $\gamma_t(T'')$-set can be extended to a TDS of T^* by adding to it the neighbor of v in T^* and the vertex at distance 2 from v in T^*. Thus, $\gamma_t(T^*) \leqslant \gamma_t(T'') + 2$. Hence, as in Case 1.1 of our proof, $\gamma_t(T) \geqslant \gamma_t(T^*)$.

Case 1.3: $e, f \in E(T) - E(T')$.

By Lemma 10, there is a $\gamma_t(T')$-set S' containing y. Thus S' can be extended to a TDS of T^* by adding to it the neighbor of v in T^* and the vertex at distance 2 from v in T^*. Thus, $\gamma_t(T^*) \leqslant \gamma_t(T') + 2 = \gamma_t(T)$, as desired.

Case 2: T is obtained from T' by operation $\mathscr{T}_2$.

The proof of Case 2 is very similar to that of the previous case and is therefore omitted. □

5.2. Necessity of Theorem 6

We proceed with a proof of the necessity condition in Theorem 6 by first presenting a lemma, the proof of which is straightforward and is therefore omitted.

Lemma 13. *If T is the tree obtained from a tree T' of order at least two by adding a star $K_{1,k}$ where $k \geqslant 1$, subdividing every edge of the star twice, and adding an edge joining the center of the star to a vertex of T', then $\gamma_t(T) = \gamma_t(T') + 2k$. (Note that if $k = 1$, we are attaching a path of length 4.)*

We are now in a position to present a proof of the necessity of Theorem 6. We proceed by induction on the order n of a tree T in Class 3. By Lemma 4, any two leaves of T are at distance at least five apart. Hence, $\mathrm{diam}(T) \geqslant 5$. Further, if $\mathrm{diam}(T) = 5$, then $n = 6$ and $T = P_6 \in \mathscr{F}$. Assume, then, that $n \geqslant 7$ and that all trees in Class 3 with order less than n belong to the family $\mathscr{F}$. Let T be a tree of order n in Class 3. Then, $\mathrm{diam}(T) \geqslant 6$.

We root T at a leaf r of a longest path P. Let u be the leaf on P different from r. Let v denote the parent of u, w denote the parent of v, x the parent of w, y the parent of x, and z the parent of y. By Lemma 4 and our choice of u, $\deg v = \deg w = 2$. We consider two possibilities depending on the degree of x in T.

Case 1: $\deg x = 2$.

Let $T' = T - \{u, v, w, x\}$. By Lemma 13, $\gamma_t(T) = \gamma_t(T') + 2$. We show that T' is in Class 3. Let T'' be obtained from T' by subdividing any two edges e and f, say, of T'. Then, $\gamma_t(T') \leqslant \gamma_t(T'')$. Let T^* be obtained from T by subdividing the two edges e and f. Then, T^* is obtained from T'' by attaching the path y, x, w, v, u to the vertex y of T'', and so, by Lemma 13, $\gamma_t(T^*) = \gamma_t(T'') + 2$. Since T is in Class 3, $\gamma_t(T) = \gamma_t(T^*)$. Hence, $\gamma_t(T) = \gamma_t(T') + 2 \leqslant \gamma_t(T'') + 2 = \gamma_t(T^*) = \gamma_t(T)$. Thus we must have equality throughout this inequality chain, whence $\gamma_t(T') = \gamma_t(T'')$. It follows that T' is in Class 3. By the inductive hypothesis, $T' \in \mathscr{F}$.

If y has status B or C in T', then T can be obtained from T' by operation $\mathscr{T}_2$, and so $T \in \mathscr{F}$. In particular, if y is a support vertex or a leaf in T', then, by Observation 8, y has status B or C in T', and so $T \in \mathscr{F}$. Hence we may assume that in T', $\deg y \geqslant 2$ and that no child of y is a leaf. Let $x' \in C(y) - \{x\}$.

Claim 1. *There exists a set $D' \subseteq V(T')$ containing y that totally dominates $V(T') - \{y\}$ such that $|D'| = \gamma_t(T') - 1$.*

Proof. Let T^* be obtained from T by subdividing the two edges uv and vw. Let u_v be the new vertex adjacent to u and v, and let v_w be the new vertex adjacent to v and w. Since T is in Class 3, $\gamma_t(T) = \gamma_t(T^*)$. Let D^* be a $\gamma_t(T^*)$-set. We may assume that $D^* \cap \{u, u_v, v, v_w, w, x, y\} = \{u_v, v, x, y\}$. Let $D' = D^* - \{u_v, v, x\}$. Then, $y \in D'$, D' totally dominates $V(T') - \{y\}$, and $|D'| = |D^*| - 3 = \gamma_t(T^*) - 3 = \gamma_t(T) - 3$. By Lemma 13, $\gamma_t(T) = \gamma_t(T') + 2$, and so $|D'| = \gamma_t(T') - 1$. □

Let D' be a set of vertices in T' satisfying Claim 1. Since D' is not a TDS of T', $x' \notin D'$. In particular, x' is not a support vertex. Let $w' \in C(x')$ and let $v' \in C(w')$. If v' is a leaf, then, by Lemma 4, $\deg w' = 2$ and so $x' \in D'$ (if $v' \in D'$, then simply replace v' in D' by x'), a contradiction. Hence, v' is not a leaf, and so $\deg v' = 2$ and the child of v' is a leaf. Hence we have shown that every child w' of x' has degree 2 and that the child v' of w' has degree 2 and is a support vertex. In particular, we observe that every child of x' is at distance 2 from a leaf, while x' is at distance 2 from a support vertex. Therefore, by Observation 8(e), x' and each child of x' has status A. By Observation 8(a), x' is adjacent to a vertex of status B or C in T', and so the vertex y must have status B or C in T'. Thus, $T \in \mathscr{F}$.

Case 2: $\deg x \geqslant 3$.

Let $k = \deg x - 1 \geqslant 2$. By Lemma 4, every child of x has degree 2 in T and every vertex at distance 2 from x in T_x (the maximal subtree of T rooted at x) is either a leaf or a support vertex of degree 2. If there are two leaves at distance 2 from x in T_x, then subdividing the two edges incident with these two leaves produces a tree with total domination number $\gamma_t(T) + 1$, contradicting the fact that T is in Class 3. Hence, T_x is obtained from a star $K_{1,k}$ by subdividing $k - 1$ edges twice and the remaining edge either once or twice.

Case 2.1: There is a leaf at distance 2 from x in T_x.

Then T_x is obtained from a star $K_{1,k}$ by subdividing $k - 1$ edges twice and the remaining edge exactly once. Let v' be the leaf at distance 2 from x in T_x, and let w' be the parent of v'. Let $T' = T - \{u, v, w\}$. Any $\gamma_t(T')$-set can be extended to a TDS of T by adding v and w, and so $\gamma_t(T) \leqslant \gamma_t(T') + 2$. On the other hand, let S be a $\gamma_t(T)$-set. Then, $v, w' \in S$ and we may assume that $w, x \in S$ (if, for example, $v' \in S$, then we can replace v' in S by x). Thus, $S - \{v, w\}$ is a TDS of T', and so $\gamma_t(T') \leqslant |S| - 2 = \gamma_t(T) - 2$. Thus, $\gamma_t(T') = \gamma_t(T) - 2$.

Claim 2. *T' is in Class 3.*

Proof. Let T'' be obtained from T' by subdividing any two edges e and f, say, of T'. Then, $\gamma_t(T') \leqslant \gamma_t(T'')$. Let T^* be obtained from T by subdividing the two edges e and f. Then, T^* is obtained from T'' by attaching the path x, w, v, u to the vertex x. Since T is in Class 3, $\gamma_t(T) = \gamma_t(T^*)$. We show that $\gamma_t(T'') \leqslant \gamma_t(T^*) - 2$.

Let S^* be a $\gamma_t(T^*)$-set. We can choose S^* as it contains no leaves. In particular, $v, w \in S^*$. Let $S'' = S^* - \{v, w\}$. If S'' is a TDS of T'', then $\gamma_t(T'') \leqslant |S''| = \gamma_t(T^*) - 2$, as desired. On the other hand, suppose S'' is not a TDS of T''. Since S'' totally dominates $V(T'') - \{x\}$, x is therefore not totally dominated by S''. Hence, $\{e, f\} = \{v'w', w'x\}$ and $k = 2$ (i.e., T_x is the path u, v, w, x, w', v' rooted at x). We may assume $e = v'w'$ and $f = w'x$. Let a be the new vertex resulting from subdividing the edge e.

We show now that there is a $\gamma_t(T)$-set S with $\{v, w, w', x, y, z\} \subseteq S$. Let F be obtained from T by subdividing the edges $w'x$ and xy, and let b (respectively, c) be the new vertex resulting from subdividing the edge $w'x$ (respectively, xy). Let D be a $\gamma_t(F)$-set. Then, $v, w' \in D$. We may assume that the leaves u and v' do not belong to D, and so $b, w \in D$. If $x \in D$, then we can simply replace x in D by y. If $c \in D$, then we can simply replace c in D by z. Hence we may assume $y, z \in D$ and $c, x \notin D$. But then $S = (D - \{b\}) \cup \{x\}$ is the desired $\gamma_t(T)$-set. Returning to the tree T'', $(S - \{v, w, x\}) \cup \{a\}$ is a TDS of T'', and so $\gamma_t(T'') \leqslant |S| - 2 = \gamma_t(T) - 2 = \gamma_t(T^*) - 2$, as desired. Hence we have shown that whether or not S'' is a TDS of T'', $\gamma_t(T'') \leqslant \gamma_t(T^*) - 2$.

Since $\gamma_t(T'') \leqslant \gamma_t(T^*) - 2$, we have $\gamma_t(T) = \gamma_t(T') + 2 \leqslant \gamma_t(T'') + 2 \leqslant \gamma_t(T^*) = \gamma_t(T)$. Consequently we must have equality throughout this inequality chain, whence $\gamma_t(T') = \gamma_t(T'')$. It follows that T' is in Class 3. □

By Claim 2, T' is in Class 3. Thus by the inductive hypothesis, $T' \in \mathscr{F}$. Since the vertex x is at distance 2 from a leaf in T', Observation 8 implies that x has status A in T'. Thus, T can be obtained from T' by operation $\mathscr{T}_1$, and so $T \in \mathscr{F}$.

Case 2.2: Every leaf in T_x is at distance 3 from x.

Then, T_x is obtained from a star $K_{1,k}$ by subdividing every edge twice. Let $T' = T - T_x$. By Lemma 13, $\gamma_t(T) = \gamma_t(T') + 2k$.

We show that T' is in Class 3. Let T'' be obtained from T' by subdividing any two edges e and f, say, of T'. Then, $\gamma_t(T') \leqslant \gamma_t(T'')$. Let T^* be obtained from T by subdividing the two edges e and f. Then, T^* is obtained from T'' by adding a star $K_{1,k}$ where $k \geqslant 2$ with center x, subdividing every edge of the star twice, and adding the edge xy. By Lemma 13, $\gamma_t(T^*) = \gamma_t(T'') + 2k$. Since T is in Class 3, $\gamma_t(T) = \gamma_t(T^*)$. Hence, $\gamma_t(T) = \gamma_t(T') + 2k \leqslant \gamma_t(T'') + 2k = \gamma_t(T^*) = \gamma_t(T)$. Thus we must have equality throughout this inequality chain, whence $\gamma_t(T') = \gamma_t(T'')$. It follows that T' is in Class 3. By the inductive hypothesis, $T' \in \mathscr{F}$.

We show next that there is a $\gamma_t(T)$-set S that contains x. Let F be obtained from T by subdividing the edges uv and vw, and let a (respectively, b) be the new vertex resulting from subdividing the edge uv (respectively, vw). Let D be a $\gamma_t(F)$-set. We may assume $u \notin D$, and so $a, v \in D$. If $b \in D$, then we can simply replace b in D by x. If $w \in D$, then we can simply replace w in D by y. Hence we may assume $b, w \notin D$, and so $x \in D$. But then $S = (D - \{a\}) \cup \{w\}$ is a $\gamma_t(T)$-set that contains x, as desired. We may assume that S contains no leaf of T_x, and so S contains the k neighbors of x in T_x and the k support vertices in T_x.

Since $S - \{x\}$ is not a TDS of T, y is totally dominated by x and by no other vertex of S. Hence in T', y is not a support vertex and no child of y is a support vertex. Suppose there exists a leaf v' in $T_y - T_x$ at distance 3 from y. Let y, x', w', v' denote the y–v' path. By Lemma 4, $\deg w' = 2$, and so S can be chosen so that $x' \in S$, contradicting our earlier observation that x is the only vertex of S that totally dominates y. Hence, there is no leaf in $T_y - T_x$ at distance 3 from y. Suppose there is a leaf u' in $T_y - T_x$ at distance 4 from y. Let y, x', w', v', u' denote the y–u' path. We may assume then that $\deg x' = \ell + 1 \geqslant 3$ (for otherwise, as in Case 1 we can show that $T \in \mathscr{F}$), and so $T_{x'}$ is obtained from a star $K_{1,\ell}$ by subdividing every edge twice.

We have shown that $T' \in \mathscr{F}$ and that either y is a leaf in T' or that y has a child x' in T' of degree $\ell + 1 \geqslant 3$ such that $T_{x'}$ is obtained from a star $K_{1,\ell}$ by subdividing every edge twice. If y is a leaf in T', then by Observation 8(d), y has status C in T'. On the other hand if y is not a leaf in T', then every child of x' is at distance 2 from a leaf, while x' is at distance 2 from a support vertex. Thus by Observation 8(e), x' and each child of x' have status A. By Observation 8(a), x' is adjacent to a vertex of status B or C in T', and so the vertex y must have status B or C in T'. In both cases, we have shown that y has status B or C in T'. Thus, T can be obtained from T' by applying operation $\mathscr{T}_2$ and then by repeated applications of operation $\mathscr{T}_1$. Hence, $T \in \mathscr{F}$. □

6. Summary

In this paper we have provided a constructive characterization of all trees in Class 3, thereby extending the result of Theorem 5. We have yet to provide a constructive characterization of all trees in Class 1 or in Class 2. We close with a characterization of caterpillars in Class 1, a proof of which is technical but routine and is therefore omitted. As a consequence of Theorems 5 and 14, caterpillars in each of the three classes are now characterized.

Theorem 14. *A caterpillar with code C is in Class* 1 *if and only if C contains consecutive nonzero entries or* $z_i \equiv 0 \pmod 4$ *or* $z_i \equiv 1 \pmod 4$ *for some* $1 \leqslant i \leqslant k$.

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